Analysis of Variance

Chapter 12

Learning Objectives

LO 12-1 List the characteristics of the *F* distribution and locate values in an *F* table.

LO 12-2 Perform a test of hypothesis to determine whether the variances of two populations are equal.

LO 12-3 Describe the ANOVA approach for testing differences in sample means.

LO 12-4 Organize data into appropriate ANOVA tables for analysis.

LO 12-5 Conduct a test of hypothesis among three or more treatment means and describe the results.

LO 12-6 Develop confidence intervals for the difference in treatment means and interpret the results.

LO 12-1 List the characteristics of the *F* distribution and locate values in an *F* table.

The *F* Distribution

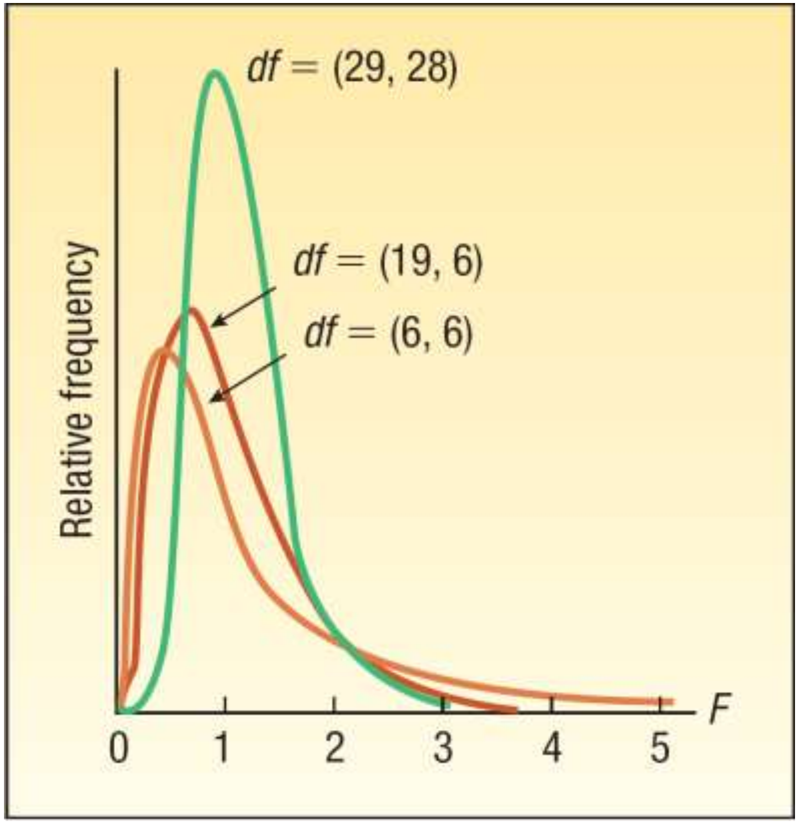
- It was named to honor Sir Ronald Fisher, one of the founders of modern-day statistics.
- Uses of the *F* distribution
  - Test whether two-samples are from populations having equal variances.
  - Compare several population means simultaneously. The test is called analysis of variance (ANOVA).
  - In both of these situations, the populations must follow a normal distribution, and the data must be at least interval-scale.

Characteristics of *F* Distribution

1. There is a "family" of *F* distributions. A member of the family is determined by two parameters: *d.f.* in the numerator and *d.f.* in the denominator.

2. The distribution is continuous, positive, and is positively skewed.

3. Asymptotic.



LO 12-2 Perform a test of hypothesis to determine whether the variances of two populations are equal.

Comparing Two Population Variances

The *F* distribution is used to test the hypothesis that the variance of one normal population equals the variance of another normal population.

Examples:

- Two Barth shearing machines are set to produce steel bars of the same length. The bars, therefore, should have the same mean length. We want to ensure that in addition to having the same mean length they also have similar variation.
- The mean rate of return on two types of common stock may be the same, but there may be more variation in the rate of return in one than the other. A sample of 10 technology and 10 utility stocks shows the same mean rate of return, but there is likely more variation in the Internet stocks.
- A study by the marketing department for a large newspaper found that men and women spent about the same amount of time per day reading the paper. However, the same report indicated there was nearly twice as much variation in time spent per day among the men than the women.

Test for Equal Variances

$$H_0: \sigma_1^2 = \sigma_2^2$$
$$H_1: \sigma_1^2 \neq \sigma_2^2$$

To conduct the test, we select a random sample of n_1 observations from one population, and a random sample of n_2 observations from the second population. The test statistic is defined as follows.

TEST STATISTIC FOR COMPARING TWO VARIANCES

$$F = \frac{s_1^2}{s_2^2} \quad [12\text{–}1]$$

Test for Equal Variances – Example

Lammers Limos offers limousine service from the city hall in Toledo, Ohio, to Metro Airport in Detroit. The president of the company is considering two routes. One is via U.S. 25 and the other via I-75. He wants to study the time it takes to drive to the airport using each route and then compare the results. He collected the following sample data, which is reported in minutes.

Using the .10 significance level, **is there a difference in the variation** in the driving times for the two routes?

U.S. Route 25	Interstate 75
52	59
67	60
56	61
45	51
70	56
54	63
64	57
	65

Test for Equal Variances – Example

Step 1: The hypotheses are:

H_0: $\sigma_1^2 = \sigma_2^2$

H_1: $\sigma_1^2 \neq \sigma_2^2$

Step 2: The significance level is 0.10.

Step 3: The test statistic is the *F* distribution.

Test for Equal Variances – Example

Step 4: State the decision rule.

Reject H_0 if computed F > critical F

TABLE 12–1 Critical Values of the F Distribution, $\alpha = .05$

Degrees of Freedom for Denominator	Degrees of Freedom for Numerator			
	5	6	7	8
1	230	234	237	239
2	19.3	19.3	19.4	19.4
3	9.01	8.94	8.89	8.85
4	6.26	6.16	6.09	6.04
5	5.05	4.95	4.88	4.82
6	4.39	4.28	4.21	4.15
7	3.97	3.87	3.79	3.73
8	3.69	3.58	3.50	3.44
9	3.48	3.37	3.29	3.23
10	3.33	3.22	3.14	3.07

Test for Equal Variances – Example

Step 5: Compute the sample standard deviations (manual method).

U.S. Route 25

$$\bar{X} = \frac{\Sigma X}{n} = \frac{408}{7} = 58.29 \qquad s = \sqrt{\frac{\Sigma(X - \bar{X})^2}{n - 1}} = \sqrt{\frac{485.43}{7 - 1}} = 8.9947$$

Interstate 75

$$\bar{X} = \frac{\Sigma X}{n} = \frac{472}{8} = 59.00 \qquad s = \sqrt{\frac{\Sigma(X - \bar{X})^2}{n - 1}} = \sqrt{\frac{134}{8 - 1}} = 4.3753$$

Test for Equal Variances – Example

Step 5: Compute the sample standard deviations (using Excel).

	A	B	C	D	E
1					
2		**US Route 75**	**I-75**		
3		52	59		
4		67	60		
5		56	61		
6		45	51		
7		70	56		
8		54	63		
9		64	57		
10			65		
11					
12	Stdev	8.995	4.375		
13					
14					

=STDEV(C3:C10)

=STDEV(B3:B9)

Test for Equal Variances – Example

Step 5: Compute the value of *F*.

U.S. Route 25

$$\bar{X} = \frac{\Sigma X}{n} = \frac{408}{7} = 58.29 \qquad s = \sqrt{\frac{\Sigma(X - \bar{X})^2}{n-1}} = \sqrt{\frac{485.43}{7-1}} = 8.9947$$

Interstate 75

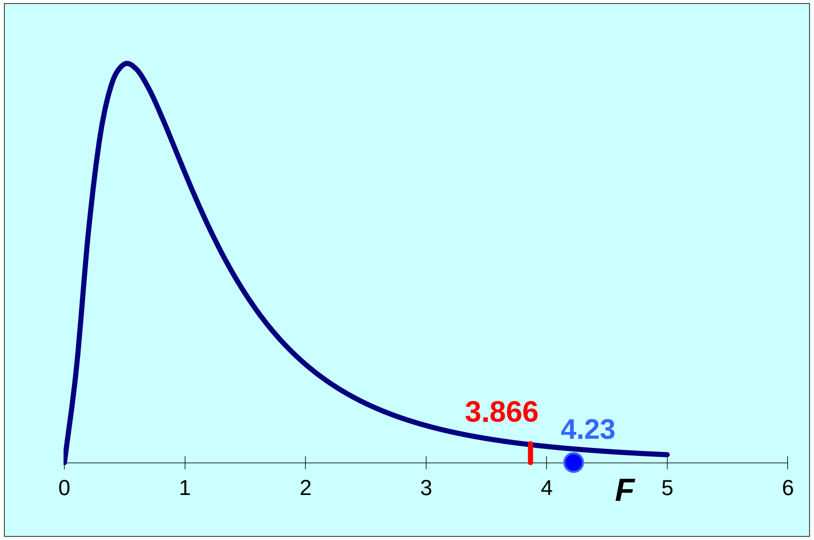
$$\bar{X} = \frac{\Sigma X}{n} = \frac{472}{8} = 59.00 \qquad s = \sqrt{\frac{\Sigma(X - \bar{X})^2}{n-1}} = \sqrt{\frac{134}{8-1}} = 4.3753$$

$$F = \frac{s_1^2}{s_2^2} = \frac{(8.9947)^2}{(4.3753)^2} = 4.23$$

Test for Equal Variances – Example

Step 6: Apply the decision rule:

Reject H_0 if computed F > critical F

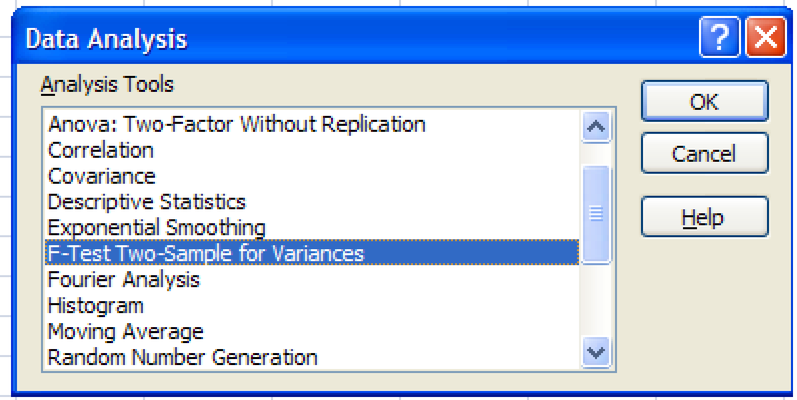


The decision is to **reject the null hypothesis**, because the computed *F* value (4.23) is larger than the critical value (3.87).

We conclude that **there is a difference in the variation** of the travel times along the two routes.

Test for Equal Variances – Excel Example

Data > Data Analysis > *F*-Test Two-Sample for Variances



num 1 variance test

	A	B	C	D	E	
1	U. S. 25	Interstate 75		F-Test Two-Sample for Variances		
2	52	59			*U. S. 25*	*Interstate 75*
3	67	60		Mean	58.29	59.00
4	56	61		Variance	80.90	19.14
5	45	51		Observations	7.00	8.00
6	70	56		df	6.00	7.00
7	54	63		F	4.23	
8	64	57		P(F<=f) one-tail	0.04	
9		65		F Critical one-tail	3.87	
10						
11						

LO 12-3 Describe the ANOVA approach for testing differences in sample means.

Comparing Means of Two or More Populations

The *F* distribution is also used for testing whether two or more sample means came from the same or equal populations.

Assumptions:

- The sampled populations follow the normal distribution.
- The populations have equal standard deviations.
- The samples are randomly selected and are independent.

LO 12-5 Conduct a test of hypothesis among three or more treatment means and describe the results.

Comparing Means of Two or More Populations

- The **null hypothesis** is that the population means are all the same. The **alternative hypothesis** is that at least one of the means is different.
- The **test statistic** is the *F* distribution.
- The hypothesis setup and decision rule:

H_0: $\mu_1 = \mu_2 = \ldots = \mu_k$

H_1: The means are not all equal

Reject H_0 if computed $F >$ critical F

α – significance level

d.f. numerator = $k - 1$

d.f. denominator = $n - k$

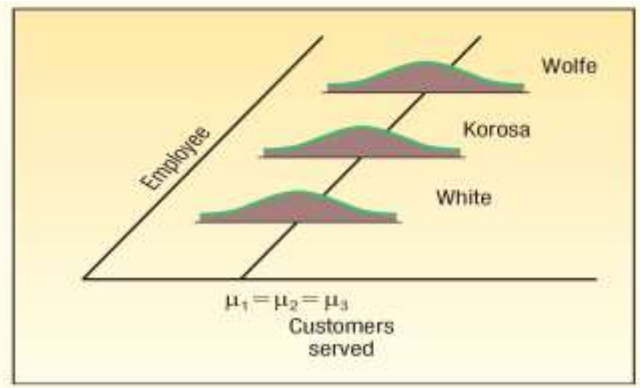
Analysis of Variance – *F-statistic*

- If there are *k* populations being sampled, the numerator degrees of freedom is $k - 1$.
- If there are a total of *n* observations, the denominator degrees of freedom is $n - k$.
- The test statistic is computed by:

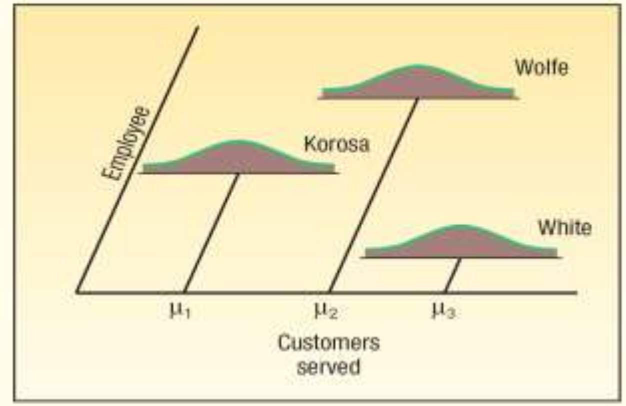
$$F = \frac{SST/(k-1)}{SSE/(n-k)}$$

Comparing Means of Two or More Populations – Illustrative Example

Joyce Kuhlman manages a regional financial center. She wishes to compare the productivity, as measured by the number of customers served, among three employees. Four days are randomly selected and the number of customers served by each employee is recorded.



Case Where Treatment Means Are the Same



Case Where Treatment Means Are Different

Comparing Means of Two or More Populations – Example

Recently a group of four major carriers joined in hiring Brunner Marketing Research, Inc., to survey recent passengers regarding their level of satisfaction with a recent flight. The survey included questions on ticketing, boarding, in-flight service, baggage handling, pilot communication, and so forth.

Twenty-five questions offered a range of possible answers: excellent, good, fair, or poor. A response of excellent was given a score of 4, good a 3, fair a 2, and poor a 1. These responses were then totaled, so the total score was an indication of the satisfaction with the flight. Brunner Marketing Research, Inc., randomly selected and surveyed passengers from the four airlines.

Northern	WTA	Pocono	Branson
94	75	70	68
90	68	73	70
85	77	76	72
80	83	78	65
	88	80	74
		68	65
		65	

Is there a difference in the mean satisfaction level among the four airlines?
Use the .01 significance level.

Comparing Means of Two or More Populations – Example

Step 1: State the null and alternate hypotheses.

H_0: $\mu_N = \mu_W = \mu_P = \mu_B$
H_1: The means are not all equal
Reject H_0 if computed F > critical F

Step 2: State the level of significance.
The .01 significance level is stated in the problem.

Comparing Means of Two or More Populations – Example

Step 3: Find the appropriate test statistic.
Because we are comparing means of more than two groups, use the *F*-statistic.

Step 4: State the decision rule.

Reject H_0 if computed F > critical F

Parameters for Critical F

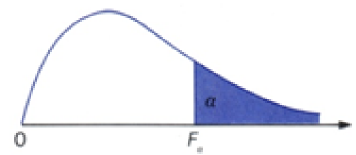
significance level = 0.01

d.f. numerator = k – 1 = 4 – 1 = 3

d.f. denominator = n – k = 22 – 4 =18

Comparing Means of Two or More Populations – Example

***F*-distribution**



Area in the right tail = 0.01

d.f. of denominator (V_2)	Degrees of Freedom = Numerator (V_1) 1	2	3	4	5	6	7	8	9	10	12
1	4052.18	4999.50	5403.35	5624.58	5763.65	5858.99	5928.36	5981.07	6022.47	6055.85	6106.32
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46
18	8.29	6.01	$\boxed{5.09}$	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17

LO 12-4

Comparing Means of Two or More Populations – Example

Step 5: Compute the value of *F* and make a decision.

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Treatments	SST	$k - 1$	$SST/(k - 1) = MST$	MST/MSE
Error	SSE	$n - k$	$SSE/(n - k) = MSE$	
Total	SS total	$n - 1$		

$$\text{SS total} = \Sigma(X - \bar{X}_G)^2$$

where:

X is each sample observation.

$\bar{X}_G$ is the overall or grand mean.

$$SSE = \Sigma(X - \bar{X}_c)^2$$

where:

$\bar{X}_c$ is the sample mean for treatment c.

Comparing Means of Two or More Populations – Example

	Northern	WTA	Pocono	Branson	Total
	94	75	70	68	
	90	68	73	70	
	85	77	76	72	
	80	83	78	65	
		88	80	74	
			68	65	
			65		
Column total	349	391	510	414	1,664
n	4	5	7	6	22
Mean	87.25	78.20	72.86	69.00	75.64

Grand Mean

$$\bar{X}_G = \frac{1{,}664}{22} = 75.64$$

Computing SS Total and SSE

$(X - \bar{X}_G)^2$

Northern	WTA	Pocono	Branson
18.36	−0.64	−5.64	−7.64
14.36	−7.64	−2.64	−5.64
9.36	1.36	0.36	−3.64
4.36	7.36	2.36	−10.64
	12.36	4.36	−1.64
		−7.64	−10.64
		−10.64	

$(X - \bar{X}_G)^2$

	Northern	WTA	Pocono	Branson	Total
	337.09	0.41	31.81	58.37	
	206.21	58.37	6.97	31.81	
	87.61	1.85	0.13	13.25	
	19.01	54.17	5.57	113.21	
		152.77	19.01	2.69	
			58.37	113.21	
			113.21		
Total	649.92	267.57	235.07	332.54	1,485.10

SS Total

$\text{SS total} = \Sigma(X - \bar{X}_G)^2$

$(X - \bar{X}_c)^2$

Northern	WTA	Pocono	Branson
6.75	−3.2	−2.86	−1
2.75	−10.2	0.14	1
−2.25	−1.2	3.14	3
−7.25	4.8	5.14	−4
	9.8	7.14	5
		−4.86	−4
		−7.86	

$(X - \bar{X}_c)^2$

	Northern	WTA	Pocono	Branson	Total
	45.5625	10.24	8.18	1	
	7.5625	104.04	0.02	1	
	5.0625	1.44	9.86	9	
	52.5625	23.04	26.42	16	
		96.04	50.98	25	
			23.62	16	
			61.78		
Total	110.7500	234.80	180.86	68	594.41

SSE

$\text{SSE} = \Sigma(X - \bar{X}_c)^2$

Computing SST

SST = SS total − SSE = 1,485.09 − 594.41 = 890.68.

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	*F*
Treatments	SST	$k - 1$	$\text{SST}/(k - 1) = \text{MST}$	MST/MSE
Error	SSE	$n - k$	$\text{SSE}/(n - k) = \text{MSE}$	
Total	SS total	$n - 1$		

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	*F*
Treatments	890.69	3	296.90	8.99
Error	594.41	18	33.02	
Total	1,485.10	21		

Setting Up the ANOVA Table

Decision Rule: Reject H_0 if computed F > critical F

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Treatments	890.69	3	296.90	8.99
Error	594.41	18	33.02	
Total	1,485.10	21		

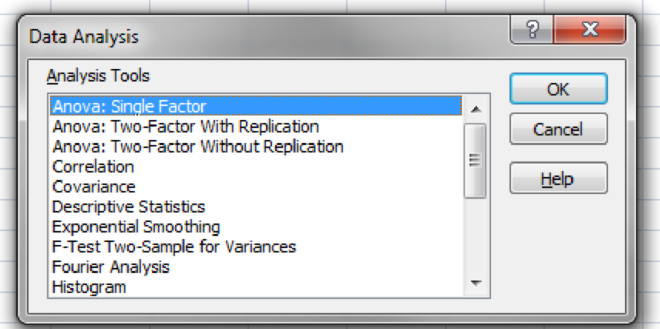
Computed F = 8.99
Critical F = 5.09

Conclude: Reject the null hypothesis (H_0: means are all equal)

Excel

Data > Data Analysis > ANOVA: Single Factor

Northern	WTA	Pocono	Branson
94	75	70	68
90	68	73	70
85	77	76	72
80	83	78	65
	88	80	74
		68	65
		65	



Anova: Single Factor

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
Northern	4	349	87.25	36.92
WTA	5	391	78.20	58.70
Pocono	7	510	72.86	30.14
Branson	6	414	69.00	13.60

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	890.68	3	296.895	8.991	0.001	5.092
Within Groups	594.41	18	33.023			
Total	1485.09	21				

LO 12-6 Develop confidence intervals for the difference in treatment means and interpret the results.

Confidence Interval for the Difference Between Two Means

When we reject the null hypothesis that the means are equal, we may want to know which treatment means differ. One of the simplest procedures is through the use of confidence intervals.

$$(\bar{X}_1 - \bar{X}_2) \pm t\sqrt{MSE\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

Where:

$\bar{X}_1$ is the mean of the first sample.
$\bar{X}_2$ is the mean of the second sample.
t is obtained from Appendix B.2. The degrees of freedom is equal to $n - k$.
MSE is the mean square error term obtained from the ANOVA table [SSE/($n - k$)].
n_1 is the number of observations in the first sample.
n_2 is the number of observations in the second sample.

Confidence Interval for the Difference Between Two Means – Example

From the previous example, develop a 95% confidence interval for the difference in the mean ratings between Northern and Branson.

Can we conclude that there is a difference between the two airlines' ratings?

Confidence Interval for the Difference Between Two Means – Example

$$= (\bar{x}_N - \bar{x}_B) \pm t\sqrt{MSE\left(\frac{1}{n_N} + \frac{1}{n_B}\right)}$$

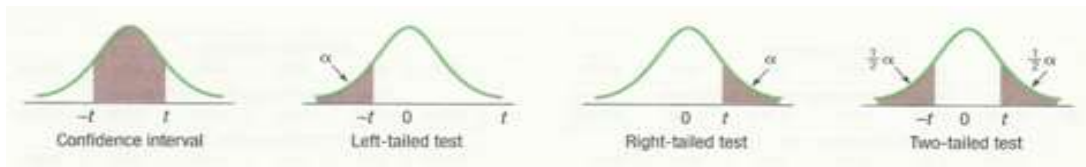
$$= (87.25 - 69.00) \pm 2.101\sqrt{33.0\left(\frac{1}{4} + \frac{1}{6}\right)}$$

$$= 18.25 \pm 7.79$$

$$(10.46, 26.04)$$

The 95% confidence interval ranges from 10.46 up to 26.04.
Conclusion: Both endpoints are positive.
Interpretation: Conclude these treatment means differ significantly.

Finding t Value for 95% Confidence Level, $d.f.$ = 18



d.f.	Confidence Interval, c					
	80.0%	90.0%	95.0%	98.0%	99.0%	99.9%
	Level of Significance for One-tailed Tests, α					
	0.100	0.050	0.025	0.010	0.005	0.001
	Level of Significance for Two-tailed Tests, α					
	0.200	0.100	0.050	0.020	0.010	0.001
1	3.078	6.314	12.706	31.821	63.657	636.619
2	1.886	2.920	4.303	6.965	9.925	31.599
3	1.638	2.353	3.182	4.541	5.841	12.924
4	1.533	2.132	2.776	3.747	4.604	8.610
5	1.476	2.015	2.571	3.365	4.032	6.869
6	1.440	1.943	2.447	3.143	3.707	5.959
7	1.415	1.895	2.365	2.998	3.499	5.408
8	1.397	1.860	2.306	2.896	3.355	5.041
9	1.383	1.833	2.262	2.821	3.250	4.781
10	1.372	1.812	2.228	2.764	3.169	4.587
11	1.363	1.796	2.201	2.718	3.106	4.437
12	1.356	1.782	2.179	2.681	3.055	4.318
13	1.350	1.771	2.160	2.650	3.012	4.221
14	1.345	1.761	2.145	2.624	2.977	4.140
15	1.341	1.753	2.131	2.602	2.947	4.073
16	1.337	1.746	2.120	2.583	2.921	4.015
17	1.333	1.740	2.110	2.567	2.898	3.965
18	1.330	1.734	$\boxed{2.101}$	2.552	2.878	3.922
19	1.328	1.729	2.093	2.539	2.861	3.883
20	1.325	1.725	2.086	2.528	2.845	3.850

Confidence Interval for the Difference Between Two Means – Minitab Output

From the previous example, develop a 95% confidence interval for the difference in the mean between American and US Airways. Can we conclude that there is a difference between the two airlines' ratings?

Approximate results can also be obtained directly from the Minitab output.

```
                                   Individual 95% CIs For Mean Based on Pooled StDev
Level     N    Mean   StDev    +---------+---------+---------+---------
Northern  4  87.250   6.076                        (-------*--------)
WTA       5  78.200   7.662              (-------*------)
Pocono    7  72.857   5.490        (------*------)
Branson   6  69.000   3.688    (------*------)
                               +---------+---------+---------+---------
                             64.0      72.0      80.0      88.0
```